Some Problems in Automata Theory Which Depend on the Models of Set Theory

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Abstract

We prove that some fairly basic questions on automata reading infinite words depend on the models of the axiomatic system **ZFC**. It is known that there are only three possibilities for the cardinality of the complement of an ω-language $L(\mathcal{A})$ accepted by a Büchi 1-counter automaton $\mathcal{A}$. We prove the following surprising result: there exists a 1-counter Büchi automaton $\mathcal{A}$ such that the cardinality of the complement $L(\mathcal{A})^-$ of the ω-language $L(\mathcal{A})$ is not determined by **ZFC**:

(1). There is a model V_1 of **ZFC** in which $L(\mathcal{A})^-$ is countable.
(2). There is a model V_2 of **ZFC** in which $L(\mathcal{A})^-$ has cardinal $2^{\aleph_0}$.
(3). There is a model V_3 of **ZFC** in which $L(\mathcal{A})^-$ has cardinal $\aleph_1$ with $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.

We prove a very similar result for the complement of an infinitary rational relation accepted by a 2-tape Büchi automaton $\mathcal{B}$. As a corollary, this proves that the Continuum Hypothesis may be not satisfied for complements of 1-counter ω-languages and for complements of infinitary rational relations accepted by 2-tape Büchi automata.

We infer from the proof of the above results that basic decision problems about 1-counter ω-languages or infinitary rational relations are actually located at the **third level** of the analytical hierarchy. In particular, the problem to determine whether the complement of a 1-counter ω-language (respectively, infinitary rational relation) is countable is in $\Sigma^1_3 \setminus (\Pi^1_2 \cup \Sigma^1_2)$. This is rather surprising if compared to the fact that it is **decidable** whether an infinitary rational relation is countable (respectively, uncountable).

Keywords: Automata and formal languages; logic in computer science; computational complexity; infinite words; ω-languages; 1-counter automaton; 2-tape automaton; cardinality problems; decision problems; analytical hierarchy; largest thin effective coanalytic set; models of set theory; independence from the axiomatic system **ZFC**.

1 Introduction

In Computer Science one usually considers either finite computations or infinite ones. The infinite computations have length ω, which is the first infinite ordinal. The theory of automata reading infinite words, which is closely related to infinite games, is now a rich theory which is used for the specification and verification of non-terminating systems, see [GTW02, PP04].

Connections between Automata Theory and Set Theory have arosen in the study of monadic theories of well orders. For example, Gurevich, Magidor and Shelah proved in [GMS83] that the monadic theory of ω_2, where ω_2 is the second uncountable cardinal, may have different complexities depending on the actual model of **ZFC** (the commonly accepted axiomatic framework for Set

Theory in which all usual mathematics can be developped), and the monadic theory of ω_2 is in turn closely related to the emptiness problem for automata reading transfinite words of length ω_2. Another example is given by [Nee08], in which Neeman considered some automata reading much longer transfinite words to study the monadic theory of some larger uncountable cardinal.

However, the cardinal ω_2 is very large with respect to ω, and therefore the connections between Automata Theory and Set Theory seemed very far from the practical aspects of Computer Science. Indeed one usually thinks that the finite or infinite computations appearing in Computer Science are "well defined" in the axiomatic framework of mathematics, and thus one could be tempted to consider that a property on automata is either true or false and that one has not to take care of the different models of Set Theory (except perhaps for the Continuum Hypothesis **CH** which is known to be independent from **ZFC**).

In [Fin09a] we have recently proved a surprising result: the topological complexity of an ω-language accepted by a 1-counter Büchi automaton, or of an infinitary rational relation accepted by a 2-tape Büchi automaton, is not determined by the axiomatic system **ZFC**. In particular, there is a 1-counter Büchi automaton $\mathcal{A}$ (respectively, a 2-tape Büchi automaton $\mathcal{B}$) and two models $\mathbf{V}_1$ and $\mathbf{V}_2$ of **ZFC** such that the ω-language $L(\mathcal{A})$ (respectively, the infinitary rational relation $L(\mathcal{B})$) is Borel in $\mathbf{V}_1$ but not in $\mathbf{V}_2$.

We prove in this paper other surprising results, showing that some basic questions on automata reading infinite words actually depend on the models of **ZFC**. In particular, we prove the following result: there exists a 1-counter Büchi automaton $\mathcal{A}$ such that the cardinality of the complement $L(\mathcal{A})^-$ of the ω-language $L(\mathcal{A})$ is not determined by **ZFC**. Indeed it holds that:

(1). There is a model V_1 of **ZFC** in which $L(\mathcal{A})^-$ is countable.

(2). There is a model V_2 of **ZFC** in which $L(\mathcal{A})^-$ has cardinal $2^{\aleph_0}$.

(3). There is a model V_3 of **ZFC** in which $L(\mathcal{A})^-$ has cardinal $\aleph_1$ with $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.

Notice that there are only these three possibilities for the cardinality of the complement of an ω-language accepted by a Büchi 1-counter automaton $\mathcal{A}$ because the ω-language $L(\mathcal{A})$ is an analytic set and thus $L(\mathcal{A})^-$ is a coanalytic set, see [Jec02, page 488].

We prove a very similar result for the complement of an infinitary rational relation accepted by a 2-tape Büchi automaton $\mathcal{B}$. As a corollary, this proves that the Continuum Hypothesis may be not satisfied for complements of 1-counter ω-languages and for complements of infinitary rational relations accepted by 2-tape Büchi automata.

In the proof of these results, we consider the largest thin (i.e., without perfect subset) effective coanalytic subset of the Cantor space 2^ω, whose existence was proven by Kechris in [Kec75] and independently by Guaspari and Sacks. An important property of $\mathcal{C}_1$ is that its cardinal depends on the models of set theory. We use this fact and some constructions from recent papers [Fin06a, Fin06b] to infer our new results about 1-counter or 2-tape Büchi automata.

Combining the proof of the above results with Shoenfield's Absoluteness Theorem we get that basic decision problems about 1-counter ω-languages or infinitary rational relations are actually located at the **third level** of the analytical hierarchy. In particular, the problem to determine whether the complement of a 1-counter ω-language (respectively, infinitary rational relation) is countable is in $\Sigma_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$. This is rather surprising if compared to the fact that it is **decidable** whether an infinitary rational relation is countable (respectively, uncountable). As a by-product of these results we get a (partial) answer to a question of Castro and Cucker about ω-languages of Turing machines.

The paper is organized as follows. We recall the notion of counter automata in Section 2. We expose some results of Set Theory in Section 3, and we prove our main results in Section 4. Concluding remarks are given in Section 5.

Notice that the reader who is not familiar with the notion of ordinal in set theory may skip part of Section 3 and just read Theorems 3.3 and 3.5 in this section. The rest of the paper relies mainly on the set-theoretical results stated in Theorem 3.5.

2 Counter Automata

We assume the reader to be familiar with the theory of formal (ω-)languages [Tho90, Sta97]. We recall the usual notations of formal language theory.

If Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 1. The *length* of x is k, denoted by $|x|$. The *empty word* has no letter and is denoted by λ; its length is 0. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ.

The *first infinite ordinal* is ω. An ω-*word* over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where for all integers $i \geq 1$, $\quad a_i \in \Sigma$. When $\sigma = a_1 \ldots a_n \ldots$ is an ω-word over Σ, we write $\sigma(n) = a_n$, $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \lambda$.

The usual concatenation product of two finite words u and v is denoted $u.v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that:

$(u.v)(k) = u(k)$ if $k \leq |u|$, and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.

The *set of* ω-*words* over the alphabet Σ is denoted by Σ^ω. An ω-*language* V over an alphabet Σ is a subset of Σ^ω, and its complement (in Σ^ω) is $\Sigma^\omega - V$, denoted V^-.

We now recall the definition of k-counter Büchi automata which will be useful in the sequel.

Let k be an integer ≥ 1. A k-counter machine has k *counters*, each of which containing a non-negative integer. The machine can test whether the content of a given counter is zero or not. And transitions depend on the letter read by the machine, the current state of the finite control, and the tests about the values of the counters. Notice that in this model some λ-transitions are allowed. During these transitions the reading head of the machine does not move to the right, i.e. the machine does not read any more letter.

Formally a k-counter machine is a 4-tuple $\mathcal{M}$=(K, Σ, Δ, q_0), where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state, and $\Delta \subseteq K \times (\Sigma \cup \{\lambda\}) \times \{0,1\}^k \times K \times \{0, 1, -1\}^k$ is the transition relation. The k-counter machine $\mathcal{M}$ is said to be *real time* iff: $\Delta \subseteq K \times \Sigma \times \{0,1\}^k \times K \times \{0, 1, -1\}^k$, i.e. iff there are no λ-transitions.

If the machine $\mathcal{M}$ is in state q and $c_i \in \mathbf{N}$ is the content of the i^{th} counter $\mathcal{C}_i$ then the configuration (or global state) of $\mathcal{M}$ is the $(k+1)$-tuple $(q, c_1, \ldots, c_k)$.

For $a \in \Sigma \cup \{\lambda\}$, $q, q' \in K$ and $(c_1, \ldots, c_k) \in \mathbf{N}^k$ such that $c_j = 0$ for $j \in E \subseteq \{1, \ldots, k\}$ and $c_j > 0$ for $j \notin E$, if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ where $i_j = 0$ for $j \in E$ and $i_j = 1$ for $j \notin E$, then we write:

$a : (q, c_1, \ldots, c_k) \mapsto_{\mathcal{M}} (q', c_1 + j_1, \ldots, c_k + j_k)$.

Thus the transition relation must obviously satisfy:

if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ and $i_m = 0$ for some $m \in \{1, \ldots, k\}$ then $j_m = 0$ or $j_m = 1$ (but j_m may not be equal to -1).

Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an ω-word over Σ. An ω-sequence of configurations $r = (q_i, c_1^i, \ldots c_k^i)_{i\geq 1}$ is called a run of $\mathcal{M}$ on σ, starting in configuration $(p, c_1, \ldots, c_k)$, iff:

(1) $(q_1, c_1^1, \ldots c_k^1) = (p, c_1, \ldots, c_k)$

(2) for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ such that $b_i : (q_i, c_1^i, \ldots c_k^i) \mapsto_{\mathcal{M}} (q_{i+1}, c_1^{i+1}, \ldots c_k^{i+1})$ and such that either $\quad a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$

or $b_1b_2\ldots b_n\ldots$ is a finite word, prefix (i.e. initial segment) of $a_1a_2\ldots a_n\ldots$

The run r is said to be complete when $a_1a_2\ldots a_n\ldots = b_1b_2\ldots b_n\ldots$

For every such run r, $\mathrm{In}(r)$ is the set of all states entered infinitely often during r.

A complete run r of M on σ, starting in configuration $(q_0, 0, \ldots, 0)$, will be simply called "a run of M on σ".

Definition 2.1 *A Büchi k-counter automaton is a 5-tuple $\mathcal{M}=(K, \Sigma, \Delta, q_0, F)$, where $\mathcal{M}'=(K, \Sigma, \Delta, q_0)$ is a k-counter machine and $F \subseteq K$ is the set of accepting states. The ω-language accepted by $\mathcal{M}$ is: $L(\mathcal{M})=\{\sigma \in \Sigma^\omega \mid$ there exists a run r of $\mathcal{M}$ on σ such that $\mathrm{In}(r) \cap F \neq \emptyset\}$*

The class of ω-languages accepted by Büchi k-counter automata is denoted $\mathbf{BCL}(k)_\omega$. The class of ω-languages accepted by *real time* Büchi k-counter automata will be denoted **r**-$\mathbf{BCL}(k)_\omega$. The class $\mathbf{BCL}(1)_\omega$ is a strict subclass of the class $\mathbf{CFL}_\omega$ of context free ω-languages accepted by Büchi pushdown automata.

We recall now the definition of classes of the arithmetical hierarchy of ω-languages, see [Sta97]. Let X be a finite alphabet. An ω-language $L \subseteq X^\omega$ belongs to the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times X^\star$ such that:

$$L = \{\sigma \in X^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots, a_{n-1}, \sigma[a_n + 1]) \in R_L\},$$

where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq X^\omega$ belongs to the class Π_n if and only if its complement $X^\omega - L$ belongs to the class Σ_n. The class Σ^1_1 is the class of *effective analytic sets* which are obtained by projection of arithmetical sets. An ω-language $L \subseteq X^\omega$ belongs to the class Σ^1_1 if and only if there exists a recursive relation $R_L \subseteq \mathbb{N} \times \{0,1\}^\star \times X^\star$ such that:

$$L = \{\sigma \in X^\omega \mid \exists \tau(\tau \in \{0,1\}^\omega \wedge \forall n \exists m((n, \tau[m], \sigma[m]) \in R_L))\}.$$

Then an ω-language $L \subseteq X^\omega$ is in the class Σ^1_1 iff it is the projection of an ω-language over the alphabet $X \times \{0,1\}$ which is in the class Π_2. The class Π^1_1 of *effective co-analytic sets* is simply the class of complements of effective analytic sets.

Recall that a Büchi Turing machine is just a Turing machine working on infinite inputs with a Büchi-like acceptance condition, and that the class of ω-languages accepted by Büchi Turing machines is the class Σ^1_1 of effective analytic sets [CG78, Sta97]. On the oher hand, one can construct, using a classical construction (see for instance [HMU01]), from a Büchi Turing machine $\mathcal{T}$, a 2-counter Büchi automaton $\mathcal{A}$ accepting the same ω-language. Thus one can state the following proposition.

Proposition 2.2 *An ω-language $L \subseteq X^\omega$ is in the class Σ^1_1 iff it is accepted by a non deterministic Büchi Turing machine, hence iff it is in the class $\mathbf{BCL}(2)_\omega$.*

3 Some Results of Set Theory

We recall that the reader who is not familiar with the notion of ordinal in set theory may skip part of this section: the main results in this section, which will be used later in this paper, are stated in Theorems 3.3 and 3.5.

We now recall some basic notions of set theory which will be useful in the sequel, and which are exposed in any textbook on set theory, like [Jec02].

The usual axiomatic system **ZFC** is Zermelo-Fraenkel system **ZF** plus the axiom of choice **AC**. The axioms of **ZFC** express some natural facts that we consider to hold in the universe of sets. For instance a natural fact is that two sets x and y are equal iff they have the same elements. This is expressed by the *Axiom of Extensionality*:

$$\forall x \forall y\ [\ x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)\ ].$$

Another natural axiom is the *Pairing Axiom* which states that for all sets x and y there exists a set $z = \{x, y\}$ whose elements are x and y:

$$\forall x \forall y\ [\ \exists z(\forall w(w \in z \leftrightarrow (w = x \vee w = y)))]$$

Similarly the *Powerset Axiom* states the existence of the set of subsets of a set x. Notice that these axioms are first-order sentences in the usual logical language of set theory whose only non logical symbol is the membership binary relation symbol $\in$. We refer the reader to any textbook on set theory for an exposition of the other axioms of **ZFC**.

A model (**V**, $\in$) of an arbitrary set of axioms $\mathbb{A}$ is a collection **V** of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of $\mathbb{A}$. We often say " the model **V**" instead of " the model (**V**, $\in$)".

We say that two sets A and B have same cardinality iff there is a bijection from A onto B and we denote this by $A \approx B$. The relation $\approx$ is an equivalence relation. Using the axiom of choice **AC**, one can prove that any set A can be well-ordered so there is an ordinal γ such that $A \approx \gamma$. In set theory the cardinal of the set A is then formally defined as the smallest such ordinal γ.

The infinite cardinals are usually denoted by $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$ The cardinal $\aleph_\alpha$ is also denoted by ω_α, when it is considered as an ordinal. The first infinite ordinal is ω and it is the smallest ordinal which is countably infinite so $\aleph_0 = \omega$ (which could be written ω_0). There are many larger countable ordinals, such as $\omega^2, \omega^3, \ldots, \omega^\omega, \ldots \omega^{\omega^\omega}, \ldots$ The first uncountable ordinal is ω_1, and formally $\aleph_1 = \omega_1$. In the same way ω_2 is the first ordinal of cardinality greater than $\aleph_1$, and so on.

The continuum hypothesis **CH** says that the first uncountable cardinal $\aleph_1$ is equal to $2^{\aleph_0}$ which is the cardinal of the continuum. Gödel and Cohen have proved that the continuum hypothesis **CH** is independent from the axiomatic system **ZFC**, i.e., that there are models of **ZFC + CH** and also models of **ZFC +** $\neg$ **CH**, where $\neg$ **CH** denotes the negation of the continuum hypothesis, [Jec02].

Let **ON** be the class of all ordinals. Recall that an ordinal α is said to be a successor ordinal iff there exists an ordinal β such that $\alpha = \beta + 1$; otherwise the ordinal α is said to be a limit ordinal and in this case $\alpha = \sup\{\beta \in \mathbf{ON} \mid \beta < \alpha\}$.

The class **L** of *constructible sets* in a model **V** of **ZF** is defined by $\mathbf{L} = \bigcup_{\alpha \in \mathbf{ON}} \mathbf{L}(\alpha)$, where the sets $\mathbf{L}(\alpha)$ are constructed by induction as follows:

(1). $\mathbf{L}(0) = \emptyset$

(2). $\mathbf{L}(\alpha) = \bigcup_{\beta < \alpha} \mathbf{L}(\beta)$, for α a limit ordinal, and

(3). $\mathbf{L}(\alpha+1)$ is the set of subsets of $\mathbf{L}(\alpha)$ which are definable from a finite number of elements of $\mathbf{L}(\alpha)$ by a first-order formula relativized to $\mathbf{L}(\alpha)$.

If **V** is a model of **ZF** and **L** is the class of *constructible sets* of **V**, then the class **L** is a model of **ZFC + CH**. Notice that the axiom (**V=L**), which means "every set is constructible", is consistent with **ZFC** because **L** is a model of **ZFC + V=L**.

Consider now a model **V** of **ZFC** and the class of its constructible sets $\mathbf{L} \subseteq \mathbf{V}$ which is another model of **ZFC**. It is known that the ordinals of **L** are also the ordinals of **V**, but the cardinals in **V** may be different from the cardinals in **L**.

In particular, the first uncountable cardinal in **L** is denoted $\aleph_1^{\mathbf{L}}$, and it is in fact an ordinal of **V** which is denoted $\omega_1^{\mathbf{L}}$. It is well-known that in general this ordinal satisfies the inequality $\omega_1^{\mathbf{L}} \leq \omega_1$. In a model **V** of the axiomatic system **ZFC + V=L** the equality $\omega_1^{\mathbf{L}} = \omega_1$ holds, but in some other models of **ZFC** the inequality may be strict and then $\omega_1^{\mathbf{L}} < \omega_1$: notice that in this case $\omega_1^{\mathbf{L}} < \omega_1$ holds because there is actually a bijection from ω onto $\omega_1^{\mathbf{L}}$ in **V** (so $\omega_1^{\mathbf{L}}$ is countable in **V**) but no such bijection exists in the inner model **L** (so $\omega_1^{\mathbf{L}}$ is uncountable in **L**). The construction of such a model is presented in [Jec02, page 202]: one can start from a model **V** of **ZFC + V=L** and construct by forcing a generic extension **V[G]** in which $\omega_1^{\mathbf{V}}$ is collapsed to ω; in this extension the inequality $\omega_1^{\mathbf{L}} < \omega_1$ holds.

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80, LT94, Sta97, PP04]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ containing at least two letters which is called the *prefix metric* and is defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v. This metric induces on Σ^ω the usual Cantor topology in which the *open subsets* of Σ^ω are of the form $W.\Sigma^\omega$, for $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set.

Definition 3.1 *Let $P \subseteq \Sigma^\omega$, where Σ is a finite alphabet having at least two letters. The set P is said to be a perfect subset of Σ^ω if and only if :*
(1) P is a non-empty closed set, and
(2) for every $x \in P$ and every open set U containing x there is an element $y \in P \cap U$ such that $x \neq y$.

So a perfect subset of Σ^ω is a non-empty closed set which has no isolated points. It is well known that a perfect subset of Σ^ω has cardinality $2^{\aleph_0}$, i.e. the cardinality of the continuum, see [Mos80, page 66].

Definition 3.2 *A set $X \subseteq \Sigma^\omega$ is said to be thin iff it contains no perfect subset.*

The following result was proved by Kechris [Kec75] and independently by Guaspari and Sacks.

Theorem 3.3 (see [Mos80] page 247) *(**ZFC**) Let Σ be a finite alphabet having at least two letters. There exists a thin Π_1^1-set $\mathcal{C}_1(\Sigma^\omega) \subseteq \Sigma^\omega$ which contains every thin, Π_1^1-subset of Σ^ω. It is called the largest thin Π_1^1-set in Σ^ω.*

An important fact is that the cardinality of the largest thin Π_1^1-set in Σ^ω depends on the model of **ZFC**. The following result was proved by Kechris, and independently by Guaspari and Sacks, see [Kan97, page 171].

Theorem 3.4 *(**ZFC**) The cardinal of the largest thin Π_1^1-set in Σ^ω is equal to the cardinal of $\omega_1^{\mathbf{L}}$.*

This means that in a given model **V** of **ZFC** the cardinal of the largest thin Π_1^1-set in Σ^ω is equal to the cardinal *in* **V** of $\omega_1^{\mathbf{L}}$, the ordinal which plays the role of the cardinal $\aleph_1$ in the inner model **L** of constructible sets of **V**.

We can now state the following theorem which will be useful in the sequel. It follows from Theorem 3.4 and from some constructions of models of set theory due to Cohen (for (a)), Levy (for (b)) and Cohen (for (c)), see [Jec02].

Theorem 3.5

(a) There is a model $\mathbf{V}_1$ *of* **ZFC** *in which the largest thin* Π^1_1*-set in* Σ^ω *has cardinal* $\aleph_1$ *with* $\aleph_1 = 2^{\aleph_0}$.

(b) There is a model $\mathbf{V}_2$ *of* **ZFC** *in which the largest thin* Π^1_1*-set in* Σ^ω *has cardinal* $\aleph_0$*, i.e. is countable.*

(c) There is a model $\mathbf{V}_3$ *of* **ZFC** *in which the largest thin* Π^1_1*-set in* Σ^ω *has cardinal* $\aleph_1$ *with* $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.

In particular, all models of (**ZFC + V=L**) satisfy (a). The models of **ZFC** satisfying (b) are the models of (**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$).

4 Cardinality problems for ω-languages

Theorem 4.1 *There exists a real-time* 1*-counter Büchi automaton* $\mathcal{A}$ *such that the cardinality of the complement* $L(\mathcal{A})^-$ *of the* ω*-language* $L(\mathcal{A})$ *is not determined by the axiomatic system* **ZFC***:*

(1). There is a model V_1 *of* **ZFC** *in which* $L(\mathcal{A})^-$ *is countable.*

(2). There is a model V_2 *of* **ZFC** *in which* $L(\mathcal{A})^-$ *has cardinal* $2^{\aleph_0}$.

(3). There is a model V_3 *of* **ZFC** *in which* $L(\mathcal{A})^-$ *has cardinal* $\aleph_1$ *with* $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.

Proof. From now on we set $\Sigma = \{0, 1\}$ and we shall denote by $\mathcal{C}_1$ the largest thin Π^1_1-set in $\{0, 1\}^\omega = 2^\omega$.

This set $\mathcal{C}_1$ is a Π^1_1-set defined by a Π^1_1-formula ϕ, given by Moschovakis in [Mos80, page 248]. Thus its complement $\mathcal{C}_1^- = 2^\omega - \mathcal{C}_1$ is a Σ^1_1-set defined by the Σ^1_1-formula $\psi = \neg\phi$. By Proposition 2.2, the ω-language $\mathcal{C}_1^-$ is accepted by a Büchi Turing machine $\mathcal{M}$ and by a 2-counter Büchi automaton $\mathcal{A}_1$ which can be effectively constructed.

We are now going to use some constructions which were used in a previous paper [Fin06a] to study topological properties of context-free ω-languages, and which will be useful in the sequel.

Let E be a new letter not in Σ, S be an integer ≥ 1, and $\theta_S : \Sigma^\omega \to (\Sigma \cup \{E\})^\omega$ be the function defined, for all $x \in \Sigma^\omega$, by:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4) \ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}} \ldots$$

We proved in [Fin06a] that if $L \subseteq \Sigma^\omega$ is an ω-language in the class $\mathbf{BCL}(2)_\omega$ and $k = cardinal(\Sigma)+2$, $S = (3k)^3$, then one can effectively construct from a Büchi 2-counter automaton $\mathcal{A}_1$ accepting L a real time Büchi 8-counter automaton $\mathcal{A}_2$ such that $L(\mathcal{A}_2) = \theta_S(L)$.

On the other hand, it is easy to see that $\theta_S(\Sigma^\omega)^- = (\Sigma \cup \{E\})^\omega - \theta_S(\Sigma^\omega)$ is accepted by a real time Büchi 1-counter automaton. The class **r-BCL**$(8)_\omega \supseteq$ **r-BCL**$(1)_\omega$ is closed under finite union in an effective way, so $\theta_S(L) \cup \theta_S(\Sigma^\omega)^-$ is accepted by a real time Büchi 8-counter automaton $\mathcal{A}_3$ which can be effectively constructed from $\mathcal{A}_2$.

In [Fin06a] we used also another coding which we now recall. Let $K = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 9699690$ be the product of the eight first prime numbers. Let Γ be a finite alphabet; here we shall set $\Gamma = \Sigma \cup \{E\}$. An ω-word $x \in \Gamma^\omega$ is coded by the ω-word

$$h_K(x) = A.C^K.x(1).B.C^{K^2}.A.C^{K^2}.x(2).B.C^{K^3}.A.C^{K^3}.x(3).B \ldots B.C^{K^n}.A.C^{K^n}.x(n).B \ldots$$

over the alphabet $\Gamma_1 = \Gamma \cup \{A, B, C\}$, where A, B, C are new letters not in Γ. We proved in [Fin06a] that, from a real time Büchi 8-counter automaton $\mathcal{A}_3$ accepting $L(\mathcal{A}_3) \subseteq \Gamma^\omega$, one can effectively construct a Büchi 1-counter automaton $\mathcal{A}_4$ accepting the ω-language $h_K(L(\mathcal{A}_3)) \cup h_K(\Gamma^\omega)^-$.

Consider now the mapping $\phi_K : (\Gamma \cup \{A, B, C\})^\omega \to (\Gamma \cup \{A, B, C, F\})^\omega$ which is simply defined by: for all $x \in (\Gamma \cup \{A, B, C\})^\omega$,

$$\phi_K(x) = F^{K-1}.x(1).F^{K-1}.x(2) \ldots F^{K-1}.x(n).F^{K-1}.x(n+1).F^{K-1} \ldots$$

Then the ω-language $\phi_K(L(\mathcal{A}_4)) = \phi_K(h_K(L(\mathcal{A}_3)) \cup h_K(\Gamma^\omega)^-)$ is accepted by a real time Büchi 1-counter automaton $\mathcal{A}_5$ which can be effectively constructed from the Büchi 8-counter automaton $\mathcal{A}_4$, [Fin06a].

On the other hand, it is easy to see that the ω-language $(\Gamma \cup \{A, B, C, F\})^\omega - \phi_K((\Gamma \cup \{A, B, C\})^\omega)$ is ω-regular and to construct a (1-counter) Büchi automaton accepting it. Then one can effectively construct from $\mathcal{A}_5$ a real time Büchi 1-counter automaton $\mathcal{A}_6$ accepting the ω-language $\phi_K(h_K(L(\mathcal{A}_3)) \cup h_K(\Gamma^\omega)^-) \cup \phi_K((\Gamma \cup \{A, B, C\})^\omega)^-$.

To sum up: we have obtained, from a Büchi Turing machine $\mathcal{M}$ accepting the ω-language $\mathcal{C}_1^- \subseteq \Sigma^\omega = 2^\omega$, a 2-counter Büchi automaton $\mathcal{A}_1$ accepting the same ω-language, a real time Büchi 8-counter automaton $\mathcal{A}_3$ accepting the ω-language $L(\mathcal{A}_3) = \theta_S(\mathcal{C}_1^-) \cup \theta_S(\Sigma^\omega)^-$, a Büchi 1-counter automaton $\mathcal{A}_4$ accepting the ω-language $h_K(L(\mathcal{A}_3)) \cup h_K(\Gamma^\omega)^-$, and a real time Büchi 1-counter automaton $\mathcal{A}_6$ accepting the ω-language $\phi_K(h_K(L(\mathcal{A}_3)) \cup h_K(\Gamma^\omega)^-) \cup \phi_K((\Gamma \cup \{A, B, C\})^\omega)^-$. From now on we shall denote simply $\mathcal{A}_6$ by $\mathcal{A}$.

Therefore we have successively the following equalities:
$L(\mathcal{A}_1) = \mathcal{C}_1^-$,
$L(\mathcal{A}_1)^- = \mathcal{C}_1$,
$L(\mathcal{A}_3)^- = \theta_S(\mathcal{C}_1)$,
$L(\mathcal{A}_4)^- = h_K(L(\mathcal{A}_3)^-) = h_K(\theta_S(\mathcal{C}_1))$,
$L(\mathcal{A}_6)^- = \phi_K(h_K(L(\mathcal{A}_3)^-)) = \phi_K(h_K(\theta_S(\mathcal{C}_1)))$.

This implies easily that the ω-languages $L(\mathcal{A}_1)^-$, $L(\mathcal{A}_3)^-$, $L(\mathcal{A}_4)^-$, and $L(\mathcal{A}_6)^- = L(\mathcal{A})^-$ all have the same cardinality as the set $\mathcal{C}_1$, because each of the maps θ_S, h_K and ϕ_K is injective.

Thus we can infer the result stated in the theorem from the above Theorem 3.5. $\square$

The following corollary follows directly from Item (3) of Theorem 4.1.

Corollary 4.2 *It is consistent with* **ZFC** *that the Continuum Hypothesis is not satisfied for complements of* 1*-counter* ω*-languages, (hence also for complements of context-free* ω*-languages).*

Remark 4.3 *This can be compared with the fact that the Continuum Hypothesis is satisfied for regular languages of infinite trees (which are closed under complementation), proved by Niwinski in [Niw91]. Notice that this may seem amazing because from a topological point of view one can find regular tree languages which are more complex than context-free* ω*-languages, as there are regular tree languages in the class* $\Delta_2^1 \setminus \mathbf{\Sigma}_1^1 \cap \mathbf{\Pi}_1^1$ *while context-free* ω*-languages are all analytic, i.e.* $\mathbf{\Sigma}_1^1$*-sets.*

Recall that a real-time 1-counter Büchi automaton $\mathcal{C}$ has a finite description to which can be associated, in an effective way, a unique natural number called the index of $\mathcal{C}$. From now on, we shall denote, as in [Fin09b], by $\mathcal{C}_z$ the real time Büchi 1-counter automaton of index z (reading words over $\Omega = \{0, 1, A, B, C, E, F\}$).

We can now use the proofs of Theorem 3.5 and 4.1 to prove that some natural cardinality problems are actually located at the **third level** of the analytical hierarchy. The notions of analytical hierarchy on subsets of $\mathbb{N}$ and of classes of this hierarchy may be found for instance in [CC89] or in the textbook [Rog67].

Theorem 4.4

(1). $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)^- \text{ is finite } \}$ *is* Π_2^1*-complete.*

(2). $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)^- \text{ is countable } \}$ *is in* $\Sigma_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$.

(3). $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)^- \text{ is uncountable}\}$ *is in* $\Pi_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$.

Proof. Item (1) was proved in [Fin09b], and item (3) follows directly from item (2).

We now prove item (2). We first show that $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)^- \text{ is countable } \}$ is in the class Σ_3^1.

Notice first that, using a recursive bijection $b : (\mathbb{N}^\star)^2 \rightarrow \mathbb{N}^\star$, we can consider an infinite word over a finite alphabet Ω as a countably infinite family of infinite words over the same alphabet by considering, for any ω-word $\sigma \in \Omega^\omega$, the family of ω-words $(\sigma_i)_{i \geq 1}$ such that for each $i \geq 1$ the ω-word $\sigma_i \in \Omega^\omega$ is defined by $\sigma_i(j) = \sigma(b(i, j))$ for each $j \geq 1$.

We can now express "$L(\mathcal{C}_z)^-$ is countable " by the formula:

$$\exists \sigma \in \Omega^\omega \ \ \forall x \in \Omega^\omega \ [(x \in L(\mathcal{C}_z)) \text{ or } (\exists i \in \mathbb{N} \ x = \sigma_i)]$$

This is a Σ_3^1-formula because "$(x \in L(\mathcal{C}_z))$", and hence also "$[(x \in L(\mathcal{C}_z)) \text{ or } (\exists i \in \mathbb{N} \ x = \sigma_i)]$", are expressed by Σ_1^1-formulas.

We can now prove that $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)^- \text{ is countable } \}$ is neither in the class Σ_2^1 nor in the class Π_2^1, by using Shoenfield's Absoluteness Theorem from Set Theory.

Let $\mathcal{A}$ be the real-time 1-counter Büchi automaton cited in Theorem 4.1 and let z_0 be its index so that $\mathcal{A} = \mathcal{C}_{z_0}$. Assume that $\mathbf{V}$ is a model of $(\mathbf{ZFC} + \omega_1^{\mathbf{L}} < \omega_1)$. In the model $\mathbf{V}$, the integer z_0 belongs to the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)^- \text{ is countable } \}$, while in the inner model $\mathbf{L} \subseteq \mathbf{V}$, the language $L(\mathcal{C}_{z_0})^-$ has the cardinality of the continuum: thus in $\mathbf{L}$ the integer z_0 does not belong to the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)^- \text{ is countable } \}$. On the other hand, Shoenfield's Absoluteness Theorem implies that every Σ_2^1-set (respectively, Π_2^1-set) is absolute for all inner models of (ZFC), see [Jec02, page 490]. In particular, if the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)^- \text{ is countable } \}$ was a Σ_2^1-set or a Π_2^1-set then it could not be a different subset of $\mathbb{N}$ in the models $\mathbf{V}$ and $\mathbf{L}$ considered above. Therefore, the set $\{z \in \mathbb{N} \mid L(\mathcal{C}_z)^- \text{ is countable } \}$ is neither a Σ_2^1-set nor a Π_2^1-set. □

Remark 4.5 *Using an easy coding we can obtain a similar result for* 1*-counter automata reading words over* Σ*, where* Σ *is any finite alphabet having at least two letters.*

Notice that the same proof gives a partial answer to a question of Castro and Cucker. They stated in [CC89] that the problem to determine whether the complement of the ω-language accepted by a given Turing machine is countable (respectively, uncountable) is in the class Σ_3^1 (respectively, Π_3^1), and asked for the exact complexity of these decision problems.

Theorem 4.6 *The problem to determine whether the complement of the* ω*-language accepted by a given Turing machine is countable (respectively, uncountable) is in the class* $\Sigma_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$ *(respectively,* $\Pi_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$*).*

We now consider acceptance of binary relations over infinite words by 2-tape Büchi automata, firstly considered by Gire and Nivat in [GN84]. A 2-tape automaton is an automaton having two tapes and two reading heads, one for each tape, which can move asynchronously, and a finite control as in the case of a (1-tape) automaton. The automaton reads a pair of (infinite) words (u, v) where u is on the first tape and v is on the second tape, so that a 2-tape Büchi automaton $\mathcal{B}$ accepts an infinitary rational relation $L(\mathcal{B}) \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$, where Σ_1 and Σ_2 are two finite alphabets. Notice that $L(\mathcal{B}) \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$ may be seen as an ω-language over the product alphabet $\Sigma_1 \times \Sigma_2$.

We shall use a coding used in a previous paper [Fin06b] on the topological complexity of infinitary rational relations. We first recall a coding of an ω-word over the finite alphabet $\Omega = \{0, 1, A, B, C, E, F\}$ by an ω-word over the alphabet $\Omega' = \Omega \cup \{D\}$, where D is an additionnal letter not in Ω. For $x \in \Omega^\omega$ the ω-word $h(x)$ is defined by :

$$h(x) = D.0.x(1).D.0^2.x(2).D.0^3.x(3).D \ldots D.0^n.x(n).D.0^{n+1}.x(n+1).D \ldots$$

It is easy to see that the mapping h from Ω^ω into $(\Omega \cup \{D\})^\omega$ is injective. Let now α be the ω-word over the alphabet Ω' which is simply defined by:

$$\alpha = D.0.D.0^2.D.0^3.D.0^4.D \ldots D.0^n.D.0^{n+1}.D \ldots$$

The following result was proved in [Fin06b].

Proposition 4.7 ([Fin06b]) *Let $L \subseteq \Omega^\omega$ be in $\mathbf{r\text{-}BCL}(1)_\omega$ and $\mathcal{L} = h(L) \cup (h(\Omega^\omega))^-$. Then*

$$R = \mathcal{L} \times \{\alpha\} \ \bigcup \ (\Omega')^\omega \times ((\Omega')^\omega - \{\alpha\})$$

is an infinitary rational relation. Moreover one can effectively construct from a real time 1-counter Büchi automaton $\mathcal{A}$ accepting L a 2-tape Büchi automaton $\mathcal{B}$ accepting the infinitary relation R.

We can now prove our second main result.

Theorem 4.8 *There exists a 2-tape Büchi automaton $\mathcal{B}$ such that the cardinality of the complement of the infinitary rational relation $L(\mathcal{B})$ is not determined by* **ZFC**. *Indeed it holds that:*

(1). There is a model V_1 of **ZFC** *in which $L(\mathcal{B})^-$ is countable.*

(2). There is a model V_2 of **ZFC** *in which $L(\mathcal{B})^-$ has cardinal $2^{\aleph_0}$.*

(3). There is a model V_3 of **ZFC** *in which $L(\mathcal{B})^-$ has cardinal $\aleph_1$ with $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.*

Proof. Let $\mathcal{A}$ be the real time 1-counter Büchi automaton constructed in the proof of Theorem 4.1, and $\mathcal{B}$ be the 2-tape Büchi automaton which can be constructed from $\mathcal{A}$ by the above Proposition 4.7. Letting $L = L(\mathcal{A})$, the complement of the infinitary rational relation $R = L(\mathcal{B})$ is equal to $[(\Omega \cup \{D\})^\omega - \mathcal{L}] \times \{\alpha\} = h(L^-) \times \{\alpha\}$. Thus the cardinality of $R^- = L(\mathcal{B})^-$ is equal to the cardinality of the ω-language $h(L^-)$, so that the result follows from Theorem 4.1. $\square$

As in the case of ω-languages of 1-counter automata, we can now state the following result, where $\mathcal{T}_z$ is the 2-tape Büchi automaton of index z reading words over $\Omega' \times \Omega'$.

Theorem 4.9

(1). $\{z \in \mathbb{N} \mid L(\mathcal{T}_z)^-$ is finite $\}$ is Π^1_2-complete.

(2). $\{z \in \mathbb{N} \mid L(\mathcal{T}_z)^-$ is countable $\}$ is in $\Sigma^1_3 \setminus (\Pi^1_2 \cup \Sigma^1_2)$.

(3). $\{z \in \mathbb{N} \mid L(\mathcal{T}_z)^-$ is uncountable$\}$ is in $\Pi^1_3 \setminus (\Pi^1_2 \cup \Sigma^1_2)$.

Proof. Item (1) was proved in [Fin09b]. Items (2) and (3) are proved similarly to the case of ω-languages of 1-counter automata, using Shoenfield's Absoluteness Theorem. $\square$

On the other hand we have the following result.

Proposition 4.10 *It is decidable whether an infinitary rational relation $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$, accepted by a given 2-tape Büchi automaton $\mathcal{B}$, is countable (respectively, uncountable).*

Proof. Let $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega$ be an infinitary rational relation accepted by a 2-tape Büchi automaton $\mathcal{B}$. It is known that $\mathrm{Dom}(R) = \{u \in \Sigma_1^\omega \mid \exists v \in \Sigma_2^\omega \;\; (u, v) \in R\}$ and $\mathrm{Im}(R) = \{v \in \Sigma_2^\omega \mid \exists u \in \Sigma_1^\omega \;\; (u, v) \in R\}$ are regular ω-languages and that one can find Büchi automata $\mathcal{A}$ and $\mathcal{A}'$ accepting $\mathrm{Dom}(R)$ and $\mathrm{Im}(R)$, [GN84]. On the other hand Lindner and Staiger have proved that one can compute the cardinal of a given regular ω-language $L(\mathcal{A})$ (see [KL08] where Kuske and Lohrey proved that this problem is actually in the class PSPACE). But it is easy to see that the infinitary rational relation R is countable if and only if the two ω-languages $\mathrm{Dom}(R)$ and $\mathrm{Im}(R)$ are countable, thus one can decide whether the infinitary rational relation R is countable (respectively, uncountable). □

Remark 4.11 *The results given by Items (2) and (3) of Theorem 4.9 and Proposition 4.10 are rather surprising: they show that there is a remarkable gap between the complexity of the same decision problems for infinitary rational relations and for their complements, as there is a big space between the class Δ_1^0 of computable sets and the class $\Sigma_3^1 \setminus (\Pi_2^1 \cup \Sigma_2^1)$.*

5 Concluding remarks

We have proved that amazingly some basic cardinality questions on automata reading infinite words depend on the models of the axiomatic system **ZFC**.

In [Fin09a] we have proved that the topological complexity of an ω-language accepted by a 1-counter Büchi automaton, or of an infinitary rational relation accepted by a 2-tape Büchi automaton, is not determined by **ZFC**.

In [Fin10], we study some cardinality questions for Büchi-recognizable languages of infinite pictures and prove results which are similar to those we have obtained in this paper for 1-counter ω-languages and for infinitary rational relations.

The next step in this research project would be to determine which properties of automata actually depend on the models of **ZFC**, and to achieve a more complete investigation of these properties.

References

[CC89] J. Castro and F. Cucker. Nondeterministic ω-computations and the analytical hierarchy. *Journal Math. Logik und Grundlagen d. Math*, 35:333–342, 1989.

[CG78] R.S. Cohen and A.Y. Gold. ω-computations on Turing machines. *Theoretical Computer Science*, 6:1–23, 1978.

[Fin06a] O. Finkel. Borel ranks and Wadge degrees of omega context free languages. *Mathematical Structures in Computer Science*, 16(5):813–840, 2006.

[Fin06b] O. Finkel. On the accepting power of two-tape Büchi automata. In *Proceedings of the 23rd International Symposium on Theoretical Aspects of Computer Science, STACS 2006*, volume 3884 of *Lecture Notes in Computer Science*, pages 301–312. Springer, 2006.

[Fin09a] O. Finkel. The complexity of infinite computations in models of set theory. *Logical Methods in Computer Science*, 5(4:4):1–19, 2009.

[Fin09b] O. Finkel. Highly undecidable problems for infinite computations. *Theoretical Informatics and Applications*, 43(2):339–364, 2009.

[Fin10] O. Finkel. Decision problems for recognizable languages of infinite pictures. In *Studies in Weak Arithmetics, Proceedings of the International Conference 28th Weak Arithmetic Days, June 17-19, 2009*, volume 196 of *Publications of the Center for the Study of Language and Information, Lecture Notes*, pages 127–151. Stanford University, 2010.

[Gir81] F. Gire. *Relations rationnelles infinitaires*. PhD thesis, Université Paris VII, 1981.

[GMS83] Y. Gurevich, M. Magidor, and S. Shelah. The monadic theory of ω_2. *Jounal of Symbolic Logic*, 48:387–398, 1983.

[GN84] F. Gire and M. Nivat. Relations rationnelles infinitaires. *Calcolo*, pages 91–125, 1984.

[GTW02] E. Grädel, W. Thomas, and W. Wilke, editors. *Automata, Logics, and Infinite Games: A Guide to Current Research [outcome of a Dagstuhl seminar, February 2001]*, volume 2500 of *Lecture Notes in Computer Science*. Springer, 2002.

[HMU01] J. E. Hopcroft, R. Motwani, and J. D. Ullman. *Introduction to automata theory, languages, and computation*. Addison-Wesley Publishing Co., Reading, Mass., 2001. Addison-Wesley Series in Computer Science.

[Jec02] T. Jech. *Set theory, third edition*. Springer, 2002.

[Kan97] A. Kanamori. *The Higher Infinite*. Springer-Verlag, 1997.

[Kec75] A.S. Kechris. The theory of countable analytical sets. *Transactions of the American Mathematical Society*, 202:259–297, 1975.

[KL08] D. Kuske and M. Lohrey. First-order and counting theories of omega-automatic structures. *Journal of Symbolic Logic*, 73(1):129–150, 2008.

[Lan69] L.H. Landweber. Decision problems for ω-automata. *Mathematical Systems Theory*, 3(4):376–384, 1969.

[LT94] H. Lescow and W. Thomas. Logical specifications of infinite computations. In J. W. de Bakker, Willem P. de Roever, and Grzegorz Rozenberg, editors, *A Decade of Concurrency*, volume 803 of *Lecture Notes in Computer Science*, pages 583–621. Springer, 1994.

[Mos80] Y. N. Moschovakis. *Descriptive set theory*. North-Holland Publishing Co., Amsterdam, 1980.

[Nee08] I. Neeman. Finite state automata and monadic definability of singular cardinals. *Jounal of Symbolic Logic*, 73:412–438, 2008.

[Niw91] D. Niwinski. On the cardinality of sets of infinite trees recognizable by finite automata. In *Proceedings of the International Conference MFCS*, volume 520 of *Lecture Notes in Computer Science*, pages 367–376. Springer, 1991.

[PP04] D. Perrin and J.-E. Pin. *Infinite words, automata, semigroups, logic and games*, volume 141 of *Pure and Applied Mathematics*. Elsevier, 2004.

[Rog67] H. Rogers. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, New York, 1967.

[Sta86] L. Staiger. Hierarchies of recursive ω-languages. *Elektronische Informationsverarbeitung und Kybernetik*, 22(5-6):219–241, 1986.

[Sta97] L. Staiger. ω-languages. In *Handbook of formal languages, Vol. 3*, pages 339–387. Springer, Berlin, 1997.

[Tho90] W. Thomas. Automata on infinite objects. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, Formal models and semantics, pages 135–191. Elsevier, 1990.

Annexe

Proof of Thorem 3.5.

(a). In the model **L**, the cardinal of the largest thin Π^1_1-set in Σ^ω is equal to the cardinal of ω_1. Moreover the continuum hypothesis is satisfied thus $2^{\aleph_0} = \aleph_1$: thus the largest thin Π^1_1-set in Σ^ω has the cardinality $2^{\aleph_0} = \aleph_1$.

(b). Let **V** be a model of (**ZFC** + $\omega_1^{\mathbf{L}} < \omega_1$). Since ω_1 is the first uncountable ordinal in **V**, $\omega_1^{\mathbf{L}} < \omega_1$ implies that $\omega_1^{\mathbf{L}}$ is a countable ordinal in **V**. Its cardinal is $\aleph_0$, and therefore this is also the cardinal in **V** of the largest thin Π^1_1-set in Σ^ω.

(c). It suffices to show that there is a model $\mathbf{V}_3$ of **ZFC** in which $\omega_1^{\mathbf{L}} = \omega_1$ and $\aleph_1 < 2^{\aleph_0}$. Such a model can be constructed by Cohen's forcing: start from a model **V** of **ZFC + V=L** (in which $\omega_1^{\mathbf{L}} = \omega_1$) and construct by forcing a generic extension **V[G]** in which are added $\aleph_2$ (or even more) "Cohen's reals", which are in fact $\aleph_2$ subsets of ω. Notice that the cardinals are preserved under this extension (see [Jec02, page 219]), and that the constructible sets of **V[G]** are also the constructible sets of **V**, thus in the new model **V[G]** of **ZFC** we still have $\omega_1^{\mathbf{L}} = \omega_1$, but now $\aleph_1 < 2^{\aleph_0}$. □